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On the non-autonomous form of the Q_4 mapping and its relation to elliptic Painlevé equations

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Abstract
We apply the deautonomization procedure to mappings obtained by one-dimensional reductions of the Q_4 and Q_3 integrable lattice systems of Adler, Bobenko and Suris. We show that in the case of the Q_4 mapping the non-autonomous forms are elliptic discrete Painlevé equations while for Q_3 the deautonomization leads to linearizable systems.

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1. Introduction

The deautonomization of integrable autonomous systems, with the help of some integrability criterion, is probably the easiest method for the construction of their non-autonomous counterparts. In the domain of discrete Painlevé equations this is the method used *par excellence*. The starting point usually is some mapping belonging to the QRT family [1]. The rationale behind this choice is that the QRT mappings are the discrete analogues of the equations defining elliptic functions. As a matter of fact, as shown by the proponents of the mapping (see also [2]), for the 'symmetric' form of the mapping and in [3] for the 'asymmetric' one, the solution of the QRT mapping is just a sampling of an elliptic function. Thus, just as the continuous Painlevé transcendents constitute an extension of the elliptic functions to the non-autonomous domain, one expects that the deautonomization of the discrete equations defining elliptic functions will lead to discrete Painlevé equations.

The basic tool for the deautonomization procedure is an integrability criterion, which allows the selection of the integrable case among all the possible ones. In the discrete domain a most convenient approach is the one based on the singularity confinement [5] property. An advantage of the singularity confinement approach over another popular discrete integrability detector, based on the computation of the algebraic entropy [6], lies in the fact that in the former approach one can study the singularities one at a time. This is not always possible in the algebraic entropy approach. On the other hand the latter is most useful when one investigates the precise type of integrability and in particular the possibility that the system be linearizable [7]. A critique, sometimes formulated, based on the non-sufficient character of the singularity confinement criterion is inapplicable in the case of deautonomization. Indeed the non-sufficient character of singularity confinement comes from the fact that the local singularity structure of the solutions of a mapping does not suffice in order to guarantee integrability: the growth of the solutions at infinity does play a crucial role as well. However in the case of the deautonomization of *integrable* mappings, the nice behaviour, i.e. slow growth, of solutions at infinity is guaranteed by the integrable character of the autonomous system. Thus the singularity confinement approach, can be, and has extensively been, applied to the derivation of integrable non-autonomous systems. Foremost among them are the discrete Painlevé equations.

While the deautonomization procedure was traditionally limited to applications based on the QRT mapping, it appeared recently that it is possible to extend this approach to mappings which do not belong to the QRT family. In particular, in [8], we showed that mappings of the Hirota–Kimura–Yahagi (HKY) [9] type do possess non-autonomous forms. The latter were obtained by a combination of the singularity confinement and algebraic entropy approaches, in particular since many integrable instances of these mappings are linearizable.

Emboldened by the success of our approach to the deautonomization of non-QRT mappings we decided to apply it to another promising case, that of mappings obtained from the reduction of the Q_4 and Q_3 systems of Adler, Bobenko and Suris (ABS) [10]. This reduction was obtained by two of the present authors (AR and BG), in collaboration with N Joshi and T Tamizhmani [11]. As we showed in that publication, the autonomous mappings obtained were of HKY rather than QRT type. In what follows we shall present their non-autonomous forms. In particular we shall show that the deautonomization of the Q_4 mapping leads to elliptic discrete Painlevé equations while that of Q_3 produces linearizable systems.

2. Derivation of the Q_4 and Q_3 mappings

In [10] Adler, Bobenko and Suris have obtained a family of integrable lattice equations. The one dubbed Q_4 is the most general, in the sense that the remaining ones can be obtained as special limits from it. Its form is most conveniently given in the parametrization initially proposed by Hietarinta involving elliptic sines:

$$\begin{aligned}\operatorname{sn}\alpha(x_{n,m}x_{n+1,m+1}+x_{n,m+1}x_{n+1,m})-\operatorname{sn}\beta(x_{n,m}x_{n+1,m}+x_{n,m+1}x_{n+1,m+1})\\ -\operatorname{sn}(\alpha-\beta)(x_{n,m}x_{n,m+1}+x_{n+1,m}x_{n+1,m+1})\\ +\operatorname{sn}\alpha\operatorname{sn}\beta\operatorname{sn}(\alpha-\beta)(1+k^2x_{n,m}x_{n,m+1}x_{n+1,m}x_{n+1,m+1})=0.\end{aligned}\tag{2.1}$$

When one takes $k=0$ the elliptic sines become circular ones and Q_4 goes over to an equation called Q_3 in the ABS classification. In the Hietarinta parametrization it has the form

$$\begin{aligned}\sin\alpha(x_{n,m}x_{n+1,m+1}+x_{n,m+1}x_{n+1,m})-\sin\beta(x_{n,m}x_{n+1,m}+x_{n,m+1}x_{n+1,m+1})\\ -\sin(\alpha-\beta)(x_{n,m}x_{n,m+1}+x_{n+1,m}x_{n+1,m+1})+\sin\alpha\sin\beta\sin(\alpha-\beta)=0.\end{aligned}\tag{2.2}$$

In order to obtain the one-dimensional reduction of Q_4 we introduced the constraint $x_{n,m+1}=x_{n+1,m}$. This resulted in the following second-order mapping.

$$\begin{aligned}(\operatorname{sn}\alpha-\operatorname{sn}\beta)x_n(x_{n+1}+x_{n-1})-\operatorname{sn}(\alpha-\beta)\left(x_{n+1}x_{n-1}+x_n^2\right)\\ +\operatorname{sn}\alpha\operatorname{sn}\beta\operatorname{sn}(\alpha-\beta)\left(1+k^2x_n^2x_{n+1}x_{n-1}\right)=0,\end{aligned}\tag{2.3}$$

where, in view of the deautonomization, the α, β should be understood as the 'one-dimensional reductions' of the α, β of equation (2.1).

Similarly the reduction of Q_3 becomes

$$(\sin\alpha-\sin\beta)x_n(x_{n+1}+x_{n-1})-\sin(\alpha-\beta)\left(x_{n+1}x_{n-1}+x_n^2\right)+\sin\alpha\sin\beta\sin(\alpha-\beta)=0 \tag{2.4}$$

In [11] we showed that both mappings (2.3) and (2.4) are not of QRT but rather of HKY type. Indeed their invariants are $\mathcal{K}=K^2$ where K is a QRT-type invariant, namely a ratio of two expressions quadratic in x_n and x_{n+1}.

Moreover the mapping (2.4) is not just integrable but as a matter of fact linearizable. As shown in [11] it suffices to subtract (2.4) from its upshift and reduce the order of the remaining homogeneous mapping by introducing the auxiliary variable $y_n=x_{n+1}/x_n$. The mapping thus obtained was

$$y_{n+1}=\frac{a+y_n-y_{n-1}\left(y_n^2-1\right)}{y_{n-1}y_n(ay_n+1)}, \tag{2.5}$$

where

$$a=-\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}.$$

As explained in [11], (2.5) is a Gambier-type mapping [12], which can generically be written as two homographic mappings in cascade. As a matter of fact the solution of (2.4) is even simpler than what one could infer from its relation to a Gambier system. A straightforward calculation shows that the general solution of (2.4) is

$$x_n=h\cos(pn+q(-1)^n+r), \tag{2.6}$$

where q and r are free parameters while q and h are given by

$$\frac{\cos\frac{\alpha+\beta}{2}}{\cos 2q}=\frac{\cos\frac{\alpha-\beta}{2}}{\cos p}=h. \tag{2.7}$$

The mappings (2.3) and (2.4) were not the only ones obtained in [11]. Since the invariants of the mappings were the squares of some QRT invariant it was natural to wonder what are the QRT mappings associated with the latter. In the case of (2.3) we found thus the mapping

$$\begin{aligned}\operatorname{sn}\alpha\operatorname{sn}\beta\operatorname{sn}(\alpha-\beta)(\operatorname{sn}\alpha-\operatorname{sn}\beta)(x_{n+1}+x_{n-1})\left(k^2x_n^4-1\right)+S_+x_n\left(x_{n+1}x_{n-1}-x_n^2\right)\\ +S_-\operatorname{sn}\alpha\operatorname{sn}\beta x_n\left(k^2x_{n+1}x_{n-1}x_n^2-1\right)=0,\end{aligned} \tag{2.8}$$

where $S_\pm=\operatorname{sn}^2(\alpha-\beta)(k^2\operatorname{sn}^2\alpha\operatorname{sn}^2\beta-1)\pm(\operatorname{sn}\alpha-\operatorname{sn}\beta)^2$. Similarly in the case of Q_3 we obtained a mapping which, after a scaling of the dependent variable, was written as

$$(y_{n+1}y_n-1)(y_ny_{n-1}-1)=y_n^4+(a+1/a)y_n^2+1 \tag{2.9}$$

which is an autonomous limit of the q-Painlevé V equation. As expected, given its relation to (2.4), this mapping is linearizable.

3. Deautonomization of the Q_4 and Q_3 mappings

Having summarized the derivation of the mappings obtained by the reduction of the integrable lattices Q_3 and Q_4 we can now embark upon their deautonomization. We start from the mapping (2.4) which we rewrite, with some hindsight, in a slightly simplified way as

$$\cos\gamma(x_{n+1}+x_{n-1})x_n-\cos z\left(x_{n+1}x_{n-1}+x_n^2\right)+(\cos^2 z-\cos^2\gamma)\cos z=0, \tag{3.1}$$

where we have taken $\alpha = \gamma + z$ and $\beta = \gamma - z$. We shall investigate the integrability of a non-autonomous form of (3.1) using the singularity confinement criterion. We ask when x_{n+1} is independent of the value of x_{n-1}. We find readily that this happens whenever $x_n = \pm\cos z$, which leads to $x_{n+1} = \pm\cos\gamma$. Iterating (3.1) we find that $x_{n+2} = \pm\cos z$ whereupon x_{n+3} recovers the lost degree of freedom.

At this point a remark is in order. Our empirical finding [13] is that the mappings of HKY type have a singularity structure different from that of the QRT mappings. As explained in [14] the generic singularity structure of a QRT mapping is one of the paired singularities: one enters the singularity through some value and exits it through some other one. A typical example is that of d-$\mathrm{P_{II}}$ where the singularity structure is $\{\pm 1, 0, \mp 1\}$. On the other hand in the case of HKY mappings one enters the singularity through some value and exits it through the *same* one. As we just saw in the case of (3.1) the singularity structure is $\{\pm\cos z, \pm\cos\gamma, \pm\cos z\}$. Whether this singularity structure is indeed a signature of HKY mappings, and what, if any, in its deep meaning are questions which are open at this stage.

Next we ask what is the possible dependence of z and γ on the independent variable. We start by assuming that both γ and z depend on n which leads to the immediate conclusion that if we enter the singularity through $x_n = \pm\cos z_n$ we must exit it by $x_{n+2} = \pm\cos z_{n+2}$. Moreover we have $x_{n+1} = \pm\cos\gamma_n$ but the condition for x_{n+3} to recover the lost degree of freedom is $x_{n+1} = \pm\cos\gamma_{n+2}$. Thus we must have $\gamma_n = \gamma_{n+2}$, which means that γ is constant up to an even–odd parity, i.e. $\gamma_{2n} = \gamma_e$, $\gamma_{2n+1} = \gamma_o$. One still must check that these values are consistent with the equation around x_{n+1}. We put $Y_n = \cos z_n$ and obtain

$$\begin{aligned}(Y_{n+1}Y_n - \cos\gamma_e\cos\gamma_o)(Y_{n-1}Y_n - \cos\gamma_e\cos\gamma_o)\\ = Y_n^4 - (\cos^2\gamma_e + \cos^2\gamma_o)Y_n^2 + \cos^2\gamma_e\cos^2\gamma_o.\end{aligned} \tag{3.2}$$

Equation (3.2) is just equation (2.9) up to a scaling and the appropriate definition of a. As explained in [11] its general solution is $Y_n = p\cos(qn + \omega)$. One can absorb p into the scaling of x (at the price of redefining γ_e and γ_o). Substituting the general solution into (3.2) with $p = 1$ we find that $q = \gamma_e + \gamma_o$. Thus we have $Y_n = \cos(n(\gamma_e + \gamma_o) + \omega)$, which means that $z_n = n(\gamma_e + \gamma_o) + \omega$. With these values of γ and z we can check that indeed the singularity is confined. Given the fact that the independent variable enters the equation through an exponential, one could conclude that the non-autonomous form of (3.1) is a q-Painlevé equation. However this is not the case: equation (3.1) is of much simpler nature. In order to show this we analyse the equation with the tools of algebraic entropy. Starting from a initial condition we obtain the successive iterates and compute the homogeneous degree of the numerator and denominator of the iterates. We find the following succession of degrees: $0, 1, 2, 4, 6, 8, 10, \ldots$, i.e. a linear growth. Thus according to our results in [7] we expect the non-autonomous form of (3.1) to be linearizable (which was also the case for the autonomous form). The linearization of the deautonomized (3.1) will be presented in the next section.

As an interesting aside we can present here a limit of (3.1) where the coefficients are polynomial in n, instead of trigonometric. (Had (3.1) been a q-Painlevé equation, this limit would have been its difference counterpart, but in the present situation we have rather another linearizable system.). We start by taking $\omega = -\pi/2 - \epsilon n_0$, $\gamma_{e,o} = \mp\pi/2 + \epsilon\delta_{e,o}$ and introduce a new dependent variable X by $x_n = \epsilon X_n$. Taking the limit $\epsilon \to 0$ we find the equation

$$(-1)^n\delta_n(X_{n+1} + X_{n-1})X_n - Z_n\left(X_{n+1}X_{n-1} + X_n^2\right) + Z_n\left(Z_n^2 - \delta_n^2\right) = 0, \tag{3.3}$$

where $Z_n = (n - n_0)(\delta_e + \delta_o)$.

Next we turn to the mapping (2.3) which we rewrite as

$$\begin{aligned}\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma(1 - k^2\,\mathrm{sn}^4 z)x_n(x_{n+1} + x_{n-1}) - \mathrm{cn}\,z\,\mathrm{dn}\,z(1 - k^2\,\mathrm{sn}^2 z\,\mathrm{sn}^2\gamma)\left(x_{n+1}x_{n-1} + x_n^2\right)\\ + (\mathrm{cn}^2 z - \mathrm{cn}^2\gamma)\,\mathrm{cn}\,z\,\mathrm{dn}\,z\left(1 + k^2x_n^2x_{n+1}x_{n-1}\right) = 0,\end{aligned} \tag{3.4}$$

where again we have taken $\alpha = \gamma + z$ and $\beta = \gamma - z$. We use the singularity confinement criterion and ask under which condition x_{n+1} is independent of the value of x_{n-1}. We find that this may happen when x is either $x_n = \pm\mathrm{cn}\, z/\mathrm{dn}\, z$ or $x_n = \pm\mathrm{dn}\, z/(k\,\mathrm{cn}\, z)$. In this case we have $x_{n+1} = \pm\mathrm{cn}\,\gamma/\mathrm{dn}\,\gamma$ and $x_{n+1} = \pm\mathrm{dn}\,\gamma/(k\,\mathrm{cn}\gamma)$, respectively. One more iteration leads to x_{n+2} taking precisely the value of x_n and x_{n+3} recovers the lost information.

The deautonomization of (3.4) follows closely that of the (3.1) case. We enter the singularity through $x_n = \pm\mathrm{cn}\, z_n/\mathrm{dn}\, z_n$ and exit it by $x_{n+2} = \pm\mathrm{cn}\, z_{n+2}/\mathrm{dn}\, z_{n+2}$, and similarly for $x_n = \pm\mathrm{dn}\, z_n/(k\,\mathrm{cn}\, z_n)$. Moreover we have $x_{n+1} = \pm\mathrm{cn}\,\gamma_n/\mathrm{dn}\,\gamma_n$ and again the condition for x_{n+3} to recover the lost degree of freedom is $x_{n+1} = \pm\mathrm{cn}\,\gamma_{n+2}/\mathrm{dn}\,\gamma_{n+2}$. We conclude again that we must have $\gamma_n = \gamma_{n+2}$, leading to a constant γ, up to an even–odd parity, i.e. $\gamma = \gamma_{e,o}$. We turn now to the consistency equation around x_{n+1}. We introduce the variable Y_n which is equal to either $Y_n = \pm\mathrm{cn}\, z_n/\mathrm{dn}\, z_n \equiv \pm\mathrm{cd}\, z_n = \mathrm{sn}(z_n \pm K)$ or $Y_n = \pm\mathrm{dn}\, z_n/(k\,\mathrm{cn}\, z_n) \equiv \pm\mathrm{dc}\, z_n/k = \mathrm{sn}(z_n \pm K + \mathrm{i}K')$ where $K, \mathrm{i}K'$ are the two standard quarter periods of the Jacobi elliptic functions. We find in all cases an equation, of the form

$$A(Y_{n+1} + Y_{n-1})\left(k^2 Y_n^4 - 1\right) + BY_n\left(k^2 Y_n^2 Y_{n+1} Y_{n-1} - 1\right) + CY_n\left(Y_{n+1}Y_{n-1} - Y_n^2\right) = 0, \tag{3.5}$$

where A, B, C are the *same* for all four singularities. Moreover γ_e and γ_o appear symmetrically in (3.5) leading to exactly the same A, B, C for n even and odd. Thus (3.5) is a symmetric QRT mapping of the exact form (2.8). From the results of [3] we know that the solution of such a mapping is a homography of an elliptic sine. It turns out that for an equation of the form (3.5), with generic A, B, C, the homography reduces to a mere multiplication and the solution of (3.5) can be written as $Y_n = p\,\mathrm{sn}(qn + \omega; kp^2)$. However here the A, B, C have specific values, namely

$$\begin{aligned} A &= \mathrm{cn}\,\gamma_e\,\mathrm{cn}\,\gamma_o\,\mathrm{dn}\,\gamma_e\,\mathrm{dn}\,\gamma_o \\ B &= -(\mathrm{cn}^2\gamma_e\,\mathrm{dn}^2\gamma_o + \mathrm{cn}^2\gamma_o\,\mathrm{dn}^2\gamma_e) \\ C &= k^2\,\mathrm{cn}^2\gamma_e\,\mathrm{cn}^2\gamma_o + \mathrm{dn}^2\gamma_e\,\mathrm{dn}^2\gamma_o. \end{aligned}$$

Substituting the solution for Y we find that it is compatible with (3.5) and the precise values of A, B, C, provided we take $p = 1$ and $q = \gamma_e + \gamma_o$. Thus the variable z is $z_n = n(\gamma_e + \gamma_o) + \omega$. The choice $p = 1$ does not lead to a loss of generality. Indeed though the form (3.4) with given γ_o, γ_e seems to fix k, it is not really so. By an appropriate rescaling of x, which implies a corresponding rescaling of the parameter k of the Jacobi elliptic functions, it is always possible to find new values of γ_o, γ_e so that one can rewrite (3.4) with any value of k.

As a further test of integrability we have computed the degree growth of the iterates of (3.4). We found the following sequence of degrees: 0, 1, 2, 5, 8, 13, 18, 25, ... We have thus a quadratic degree growth which is compatible with the integrable character of (3.4) but which shows that (3.4) is not linearizable. Equation (3.4) is in fact an perfect example of an elliptic discrete Painlevé equation [15]. As a matter of fact this is the first instance (to the authors' knowledge, of course) that an elliptic Painlevé equation is derived as a reduction of an integrable lattice equation.

4. Integration of the linearizable mapping

As explained in the previous section, equation (3.1) and its limit (3.3) are expected to be linearizable, just as was the case for the autonomous reduction (2.4). It is thus natural that we seek their linearization. However before embarking upon these calculations we feel that some generalization of (3.1) is mandatory. It is indeed our experience that when a mapping is linearizable its coefficients can be expressed in terms of some completely arbitrary functions [16]. This is indeed the case for projective mappings as well as for the Gambier one. It turns out that the form of (3.1) cannot accommodate a free function. However it is possible to generalize it slightly and *still preserve linearizability*. We are thus going to work with the mapping

$$ax_{n+1}x_{n-1} + b(x_{n+1} + x_{n-1})x_n + cx_n^2 = 1, \tag{4.1}$$

i.e. a form similar to that of (3.1) but where the relative coefficient of the $x_{n+1}x_{n-1}$ and x_n^2 terms is no longer 1. The parameters a, b, c are now functions of the independent variable.

We are not going to go into all the details of the derivation. It suffices to say that the linearization can be obtained in terms of a Gambier mapping the form of which is inspired by the one found in the case of (2.4). As already explained in [17] we subtract (4.1) from its upshift (i.e., taking its discrete derivative) and reduce the order of the remaining homogeneous mapping by introducing the auxiliary variable $y_n = x_{n+1}/x_n$. We find the mapping

$$\begin{aligned} &b_{n+1}y_n^2y_{n+1}y_{n-1} + c_{n+1}y_n^2y_{n-1} + a_{n+1}y_ny_{n+1}y_{n-1} \\ &\qquad + (b_{n+1} - b_n)y_ny_{n-1} - a_ny_n - c_ny_{n-1} - b_n = 0. \end{aligned} \tag{4.2}$$

This mapping is again a Gambier one. Indeed it can be written as a system of two discrete Riccatis in cascade

$$y_n = \frac{\alpha + z_n(\beta + y_{n-1})}{y_{n-1}} \tag{4.3a}$$

$$z_{n+1} = -\delta - \frac{z_n}{\gamma + \kappa z_n}, \tag{4.3b}$$

where $\alpha, \beta, \gamma, \delta$ and κ are functions of the independent variable. In order to simplify the presentation of the results we introduce the (free) function $g_n = b_n/a_n$. A detailed calculation shows that it is possible to express the parameters of the Gambier mapping as follows:

$$\begin{aligned} \alpha_n &= \frac{g_{n-1}}{g_{n+1}} \\ \beta_n &= g_{n-1} \\ \gamma_n &= \frac{g_{n-1}b_{n+1}}{b_ng_n} \\ \kappa_n &= \frac{b_{n+1}g_ng_{n-1}}{b_ng_{n+1}} \\ \delta_n &= \frac{g_{n+1}}{1 + g_ng_{n+1}}\left(\frac{g_n + g_{n+2}}{g_{n+2}} - \frac{b_n}{b_{n+1}}\frac{g_{n+1} + g_{n-1}}{g_{n-1}}\right). \end{aligned}$$

Moreover the three functions a, b and c can be expressed in terms of the free function g. From the definition of g we have

$$a_n = \frac{b_n}{g_n} \tag{4.4}$$

and moreover we find

$$c_n = g_n\frac{b_ng_{n-1}g_{n-2}(g_{n+1} + g_{n-1}) + b_{n-1}g_{n+1}(1 - g_{n-1}g_{n-2})}{g_{n+1}g_{n-1}g_{n-2}(1 + g_ng_{n-1})} \tag{4.5}$$

while b is given by the linear equation

$$\begin{aligned} &b_{n+1}g_{n-1}g_{n-2}(g_{n-1}g_n + 1)(g_{n+1}g_{n+2} - 1) + b_ng_{n-2}g_{n+2}\left(g_{n-1}^2 - g_{n+1}^2\right) \\ &\qquad + b_{n-1}g_{n+1}g_{n+2}(g_{n+1}g_n + 1)(1 - g_{n-1}g_{n-2}) = 0. \end{aligned} \tag{4.6}$$

Thus equation (4.1) is linearizable and as expected its general nonautonomous form does involve a free function.

Before concluding this section it would be interesting, as an aside, to consider the degeneration of the mapping (4.1). As already shown by Adler, Bobenko and Suris, the integrable lattice Q_3 does, under the appropriate limiting procedure, degenerate to the lattice these authors of have dubbed Q_2. In [11] we have presented its reduced form

$$(x_{n+1} - x_n)(x_n - x_{n-1}) + \alpha(x_{n+1} + 2x_n + x_{n-1}) + \beta = 0 \tag{4.7}$$

and have shown that it is linearizable in the same way as the mapping obtained from the reduction of Q_3. It would be interesting to present here its deautonomization. For the linearization of the autonomous form of (4.7) we had started by subtracting it from its upshift and reducing the order of the remaining mapping by introducing the auxiliary variable $y_n = x_{n+1} - x_n$. Here we start by consider the Gambier mapping:

$$y_n = y_{n-1} z_n + g_n(z_n + 1) \tag{4.8a}$$

$$z_{n+1} z_n = \frac{f_n}{f_{n+1}}. \tag{4.8b}$$

Eliminating z and introducing the variable x we obtain a mapping which can be written as $f_{n+1} M_{n+1} - f_n M_n$, where $M_n = 0$ defines a mapping which is the nonautonomous form of (4.7). We find that f can be explicitly given in terms of the free function g:

$$f_n = \frac{\kappa g_n + 2k(-1)^n}{(g_n + g_{n-1})(g_n + g_{n+1})}, \tag{4.9}$$

where κ and k are two arbitrary constants. The mapping M has now the form

$$(x_{n+1} - x_n)(x_n - x_{n-1}) + x_{n+1} g_{n-1} + x_n(g_n - g_{n+1} + \gamma_n(g_n + g_{n-1})) + x_{n-1} g_{n+1} + \beta_n = 0, \tag{4.10}$$

where $\gamma_n = (\kappa g_{n+1} - 2k(-1)^n)/(\kappa g_n + 2k(-1)^n)$, $\beta_n = -g_{n-1} g_{n+1} + (c + k(-1)^n)/f_n$ and c is another free constant. It is clear from the expression of (4.9) that this nonautonomous form could not have been obtained by simply allowing the parameters α and β in (4.7) to depend on n.

5. Conclusion

In this paper we have examined two (in fact, three) mappings we have obtained in some previous work as reductions of the Q integrable lattice systems of Adler, Bobenko and Suris. As shown in [11] those mappings were not of QRT type and the conclusion we had drawn there was that there appeared to be no obvious way to extend them to non-autonomous forms. The present work aimed at remedying this.

The approach we adopted in the present paper was that of deautonomization, based on the singularity confinement criterion, complemented by the computation of the algebraic entropy. The key element was, in the case of the Q_4 and Q_3 mappings, the adequate parametrization which allowed the identification of the singular values of the dependent variable. This led to a non-autonomous form for the Q_4 mapping where the independent variable appeared in the argument of elliptic functions. While this is not the first example of an elliptic discrete Painlevé equation, it is the first time that such an elliptic integrable system is obtained from the reduction of an integrable lattice equation. This is also, by far, the simplest form of an elliptic discrete Painlevé equation ever found.

The case of Q_2 mapping was more challenging: its non-autonomous form was obtained from the appropriate limit of the (non-autonomous form of the) Q_3 mapping. In this case the straightforward deautonomization, i.e. allowing the parameters of the mapping to depend on the independent variable, would not have given the desired result. This should be an indication for future deautonomization investigations: in some cases one must extend the autonomous form, introducing *a priori* superfluous parameters, in order to ensure a parametrization rich enough, to be amenable to deautonomization.

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